# Nonlinear Programming and Grossone: Quadratic Programming and the role of Constraint Qualifications

Renato De Leone*

**Abstract**

A novel and interesting approach to infinite and infinitesimal numbers was recently proposed in a series of papers and a book by Y. Sergeyev. This novel numeral system is based on the use of a new infinite unit of measure (the number *grossone*, indicated by the numeral ①), the number of elements of the set, $\mathbb{N}$, of natural numbers. Based on the use of ①, De Cosmis and De Leone [1] have then proposed a new exact differentiable penalty function for constrained optimization problems. In this paper these results are specialized to the important case of quadratic problems with linear constraints. Moreover, the crucial role of Constraint Qualification conditions (well know in constraint minimization literature) is also discussed with reference to the new proposed penalty function.

Keywords: Nonlinear optimization, Grossone, Penalty Functions

In a series of papers and in a book [2, 3, 4, 5], Yaroslav Sergeyev proposed an interesting and fresh approach to infinite and infinitesimal numbers whose peculiar characteristic is the attention to numerical aspects and to applications. This novel numeral system is based on the use of a new infinite unit of measure (the numeral *grossone*, indicated by ①), the number of elements of the set, $\mathbb{N}$, of natural numbers. Grossone is introduced through the following three properties:

- *Infinity.* Any finite natural number $n$ is less than grossone, *i.e.*, $n <$ ①.
- *Identity.* The following relationships link ① to the identity elements 0 end 1

$$0 \cdot ① = ① \cdot 0 = 0,\ ① - ① = 0,\ \frac{①}{①} = 1,\ ①^0 = 1,\ 1^{①} = 1, \quad 0^{①} = 0 \quad (1)$$

- *Divisibility.* For any finite natural number $n$, the sets $\mathbb{N}_{k,n}$, $1 \le k \le n$,

$$\mathbb{N}_{k,n} = k, k+n, k+2n, k+3n, ...., \quad 1 \le k \le n, \quad \bigcup_{k=1}^{n} \mathbb{N}_{k,n} = \mathbb{N} \quad (2)$$

*School of Science and Technology, University of Camerino, Camerino, Italy. Email *renato.deleone@unicam.it*

have the same number of elements indicated by $\frac{①}{n}$.

It should be noted that ① is not a symbol and is not used to make symbolic calculation. In fact, the new numeral ① belongs to $\mathbb{N}$, and it has both cardinal and ordinal properties, exactly as the "standard", finite natural numbers. The approach proposed by Sergeyev (see in particular [5]) and employed here is more computationally and practically oriented. The same approach used in chemistry or physics and based on a strict correlation between researcher, object of investigation and tools utilized to investigate the object or the phenomenon, is proposed also in mathematics. In this context, a new positional numeral system with base ① is introduced, where the number

$$C = c_{p_m}①^{p_m} + c_{p_{m-1}}①^{p_{m-1}} + \ldots + c_{p_1}①^{p_1} + c_{p_0}①^{p_0} + c_{p_{-1}}①^{p_{-1}} + \ldots + c_{p_{-k}}①^{p_{-k}} \tag{3}$$

is represented by the record

$$c_{p_m}①^{p_m} \ldots c_{p_1}①^{p_1} c_{p_0}①^{p_0} c_{p_{-1}}①^{p_{-1}} \ldots c_{p_{-k}}①^{p_{-k}} \tag{4}$$

where

$$p_m > p_{m-1} > \ldots > p_1 > p_0 = 0 > p_{-1} > \ldots > p_{-k}.$$

This numeral system for expressing numbers are the tools for observation, and, by using a more powerful numerical system, it is possible to reach more precise results in applied and pure mathematics. The numerals $c_i \neq 0$, belonging to the "traditional" numeral system are called *grossdigits*. Numbers $p_i$ are called *grosspowers* and can be finite, infinite, and infinitesimal. The term having $p_0 = 0$ represents the finite part of $C$, the terms having finite positive grosspowers represent the infinite part of $C$, and terms having negative finite grosspowers represent the infinitesimal parts of $C$. The term $①^{-1}$ is an infinitesimal.

The new methodology is under study from both theoretical and applied viewpoints. On the one hand, many authors connected the new approach to the historical panorama of ideas dealing with infinities and infinitesimals [6, 7, 8, 9, 10, 11]. In particular, relations of the new approach to bijections are studied in [8] and metamathematical investigations on the new theory and its non-contradictory can be found in [7]. On the other hand, the new methodology has been successfully applied in many areas such as cellular automata [12, 13, 14], Euclidean and hyperbolic geometry [15, 16], percolation [17, 18, 19], fractals [20, 21, 22, 23, 19], numerical differentiation and optimization [24, 25], infinite series and the Riemann zeta function [26, 27, 28], the first Hilbert problem, Turing machines, and lexicographic ordering [29, 11, 30, 31, 32], ordinary differential equations [33, 34, 35, 36], etc. The interested reader is invited to have refer to the surveys [37, 38] and to the book [2] for a introduction for general public.

Based on this new numeral system, various applications to linear and non-linear problems have been proposed in [1]. In particular, the relationships of KKT points [39] of the constrained optimization problem

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{subject to} & g(x) \leq 0 \\ & h(x) = 0 \end{array} \tag{5}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ , and $h : \mathbb{R}^n \to \mathbb{R}^p$ and stationary points of the unconstrained problem

$$\min_x f(x) + \frac{1}{2}① \left\|\max\{0, g(x)\}\right\|^2 + \frac{1}{2}① \left\|h(x)\right\|^2 =: F(x) \tag{6}$$

have been investigated.

In this paper we discuss two different aspects of the use of ① in constrained minimization problems.

Firstly in Section 1 we discuss the importance of Constraint Qualification (whose role is well known and studied in nonlinear optimization literature) when ① is utilized to move from constrained to unconstrained problems. It is well known that exactness in (6) cannot be achieved if we use, instead of ①, any finite real value. Moreover, differentiable exact penalty functions can be constructed [40], but it is necessary to include terms related to first order optimality conditions thus making the penalty function much more complicate. The use of ① provides a very simple, but powerful alternative. However, Constraint Qualification play an fundamental role even in this context. An example shows that, when Constraint Qualifications do not hold, spurious or incorrect solutions can arise from stationary point of the unconstrained problem.

In Section 2, following [1], again we use ① to define an unconstrained minimization problem for a quadratic problem with linear constraints. Here, Constraint Qualification conditions are trivially satisfied, since all constraints are linear. We will show that the finite term of a stationary point for the unconstrained problem provides a solution of the constrained problem, while the multipliers can be easily and directly obtained from the $①^{-1}$ terms.

We briefly describe our notation now. All vectors are column vectors and will be indicated with lower case Latin letter ($x$, $z$, $\ldots$). Subscripts indicate components of a vector, while superscripts are used to identify different vectors. Matrices will be indicated with upper case roman letter ($A$, $B$, $\ldots$). The set of real numbers and the set of nonnegative real numbers will be denoted by $\mathbb{R}$ and $\mathbb{R}_+$ respectively. The rank of a matrix $A$ will be indicated by $\operatorname{rank} A$. The space of the $n$–dimensional vectors with real components will be indicated by $\mathbb{R}^n$ and $\mathbb{R}^n_+$ is an abbreviation for the nonnegative orthant in $\mathbb{R}^n$. The symbol $\|x\|$ indicates the Euclidean norm of a vector $x$. Superscript $^T$ indicates transpose. The scalar product of two vectors $x$ and $y$ in $\mathbb{R}^n$ will be denoted by $x^T y$. Here and throughout the symbols $:=$ and $=:$ denote definition of the term on the left and the right sides of each symbol, respectively. The gradient $\nabla f(x)$ of a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $x \in \mathbb{R}^n$ is assumed to be a column vector.

# 1 The importance of Constraint Qualifications

It is well know that the Karush–Kuhn–Tucker (KKT) first order optimality conditions for constrained optimization strongly depend on specific conditions on the constraints of the problem, known as Constraint Qualifications (CQs).

Since the seminal work of Kuhn and Tucker [41], a number of different (inter–related) Constraint Qualification conditions have been proposed in literature both for the case of only inequality constraints and for the more general case of equality of inequality constraints (see [42] and reference therein, including the two schemes showing the relationships between the different Constraint Qualification conditions, [43], and [39] for additional in depth details on Constraint Qualification conditions).

A direction $d \in \mathbb{R}^n$ is tangent to the set

$$X = \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0\}$$

at $\bar{x}$ if there exists a sequence $\left\{z^k\right\} \subseteq X$ with $\lim_{k\to\infty} z^k = \bar{x}$ and a sequence $\{\theta_k\}$ of positive scalars with $\lim_{k\to\infty} \theta_k = 0$ such that

$$\lim_{k\to\infty} \frac{z^k - \bar{x}}{\theta_k} = d. \tag{7}$$

The set of all tangent vectors to $X$ at $\bar{x}$ is called the tangent cone at $\bar{x}$.

The set of linearized feasible directions is the set

$$\left\{ \begin{array}{lcll} d \in \mathbb{R}^n & : & \nabla g_i(\bar{x})^T d \leq 0, & i = 1, \ldots, m, i \in \mathcal{A}(\bar{x}), \\ & & \nabla h_k(\bar{x})^T d = 0, & k = 1, \ldots, p \end{array} \right\}.$$

where $\mathcal{A}(\bar{x})$ is the set of indices active at $\bar{x}$.

Constraint Qualification conditions impose that the tangent cone and the cone of linearized directions are equal.

Given a point $\bar{x} \in X$, the set of active constraints at $\bar{x}$ is

$$I(\bar{x}) = \{1, \ldots p\} \cup \{i \in \{1, \ldots m\} : g_i(\bar{x}) = 0\}.$$

The Linear Independence Constraint Qualification (LICQ) condition holds at $\bar{x} \in X$ if the set of gradients of the active constraints at $\bar{x}$ is linearly independent.

In [1] a Modified LICQ (MLICQ) condition is introduced. More specifically, for the constrained optimization problem (5) the MLICQ are satisfied at $x \in \mathbb{R}^n$ if the vectors

$$\left\{\nabla g_i(x)_{i:g_i(x)\geq 0}, \nabla h_j(x)_{j=1,\ldots,p}\right\}$$

are linearly independent. This conditions is stronger than standard LICQ; however, gradients with indices corresponding to $g_i(x) < 0$ are not considered. Under MLICQ it can be shown that if

$$x^* = x^{*,0} + ①^{-1} x^{*,1} + ①^{-2} x^{*,2} + \ldots$$

is a stationary point for the unconstrained minimization Problem (6), then the triplet $\left(x^{*,0}, \mu^*, \pi^*\right)$ is a KKT point of (5) where

$$\mu_i^* = \left\{ \begin{array}{ll} 0 & \text{if } g_i^{*,0} < 0, \\ \max\left\{0, g_i^{*,1}\right\} & \text{if } g_i^{*,0} = 0 \end{array} \right. ,$$

$\pi^*$ is the ①$^{-1}$ term in $h(x^*)$ and $g_i^{*,0} \in \mathbb{R}$ and $g_i^{*,1} \in \mathbb{R}$ are respectively the ①$^0$ and ①$^{-1}$ finite grossdigits in $g_i(x^*)$.

In this section we want to stress the importance of Constraint Qualification conditions in the proposed approach based on ①.

Consider the constrained optimization problem [44, Example 12.1 pag. 308]

$$\begin{array}{ll} \min & x_1 + x_2 \\ \text{subject to} & x_1^2 + x_2^2 - 2 = 0 \end{array} \tag{8}$$

for which the optimal solution is $x^* = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. The Lagrangian function is

$$L(x,\pi) = x_1 + x_2 + \pi\left(x_1^2 + x_2^2 - 2\right) \tag{9}$$

and the pair $\left(x^*, \pi^* = \frac{1}{2}\right)$ satisfies the KKT conditions. Consider now the function

$$F(x) := x_1 + x_2 + \frac{①}{2}\left(x_1^2 + x_2^2 - 2\right)^2$$

and the unconstrained minimization problem

$$\min_x F(x).$$

It should be noticed that here ① is not used here as a variable as $\pi$ in (9) or an indefinite symbol. In contrast, ① is a number and, therefore, $F(x)$ is a function that can assume infinite and infinitesimal values expressed in the form (3) at points $x$ that also can be in the form (3).

The First–Order Optimality Conditions $\nabla F(x) = 0$ become

$$\left\{ \begin{array}{l} 1 + 2①x_1\left(x_1^2 + x_2^2 - 2\right) = 0 \\ \\ 1 + 2①x_2\left(x_1^2 + x_2^2 - 2\right) = 0 \end{array} \right. . \tag{10}$$

To solve (11), note that from symmetry $x_1 = x_2$. Then starting from

$$1 + 4①x\left(x^2 - 1\right) = 0 \tag{11}$$

consider a representation of $x$

$$x = A + B①^{-1} + C①^{-2} + \cdots$$

and then equate to 0 the different power of ①. In (11), equating to 0 the ① term, it follows that the

$$A(A^2 - 1) = 0$$

and hence $A$ can be 0, 1 or -1. Then, for each of these values of $A$, it is possible to derive the ①$^{-1}$ grossdigit in the representation of $x$ equating to 0 the ①$^0$ term in (11).

Therefore, a solution, neglecting the terms of order ①$^{-3}$ and beyond, is given by

$$\begin{cases} x_1^* = -1 - \frac{1}{8}①^{-1} + C①^{-2} \\ \\ x_2^* = -1 - \frac{1}{8}①^{-1} + C①^{-2} \end{cases}$$

from which

$$(x_1^*)^2 + (x_2^*)^2 - 2 = \frac{1}{2}①^{-1} + D①^{-2}$$

for some $C, D \in \mathbb{R}$. Therefore, the finite part of $x^*$ and the $①^{-1}$ of the constraint provide a KKT pair for the constrained problem.

Note that another solution of $\nabla F(x) = 0$, neglecting again the terms of order $①^{-3}$ and beyond, is given by

$$\begin{cases} \bar{x}_1 = 1 - \frac{1}{8}①^{-1} + \tilde{C}①^{-2} \\ \\ \bar{x}_2 = 1 - \frac{1}{8}①^{-1} + \tilde{C}①^{-2} \end{cases}$$

where $\tilde{C} \in \mathbb{R}$, that corresponds to a point of maximum. An additional solution is given (again we neglect the terms of order $①^{-3}$ and beyond) by

$$\begin{cases} \hat{x}_1 = \frac{1}{4}①^{-1} + \bar{C}①^{-2} \\ \\ \hat{x}_2 = \frac{1}{4}①^{-1} + \bar{C}①^{-2} \end{cases}$$

where $\bar{C} \in \mathbb{R}$. For this latter point, the finite part $\hat{x}^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ does not satisfy LICQ.

Consider now the constrained problem

$$\begin{array}{ll} \min & x_1 + x_2 \\ \text{subject to} & \left(x_1^2 + x_2^2 - 2\right)^2 = 0 \end{array} \tag{12}$$

where the only difference respect to Problem (8) is in the constraint $\left(x_1^2 + x_2^2 - 2\right)^2 = 0$ instead of simply $x_1^2 + x_2^2 - 2 = 0$. The feasible region remains the same, the optimal solution is still $x^* = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, and the tangent cone (which only depends on the feasible region and not on its representation) remains unchanged. However, in this case [44, Example 12.4 pag. 318] the cone of linearized feasible directions is different and, therefore, Constraint Qualification conditions are not satisfied.

In this case the Lagrangian function is

$$L(x, \pi) = x_1 + x_2 + \pi \left(x_1^2 + x_2^2 - 2\right)^2$$

and the KKT conditions are

$$\begin{cases} 1 - 4\pi x_1 \left(x_1^2 + x_2^2 - 2\right) = 0 \\ 1 - 4\pi x_2 \left(x_1^2 + x_2^2 - 2\right) = 0 \\ \left(x_1^2 + x_2^2 - 2\right)^2 = 0 \end{cases} \tag{13}$$

for which no solution exists.

Let now

$$F(x) := x_1 + x_2 + \frac{\textcircled{1}}{2}\left(x_1^2 + x_2^2 - 2\right)^4.$$

The First–Order Optimality Conditions are

$$\begin{cases} 1 + 4\textcircled{1}x_1\left(x_1^2 + x_2^2 - 2\right)^3 = 0, \\ 1 + 4\textcircled{1}x_2\left(x_1^2 + x_2^2 - 2\right)^3 = 0. \end{cases}$$

Let the solution of the above system be

$$x_1^* = x_2^* = A + B\textcircled{1}^{-1} + C\textcircled{1}^{-2}$$

with $A, B, \text{and}\, C \in \mathbb{R}$. Now

$$4\textcircled{1}x_1^* = 4A\textcircled{1} + 4B + 4C\textcircled{1}^{-1}$$

and

$$\begin{aligned}(x_1^*)^2 + (x_2^*)^2 - 2 &= 2\Big[A^2 + B^2\textcircled{1}^{-2} + C^2\textcircled{1}^{-4} + 2AB\textcircled{1}^{-1} + 2AC\textcircled{1}^{-2} + 2BC\textcircled{1}^{-3}\Big] - 2 \\ &\quad \big(2A^2 - 2\big) + 2AB\textcircled{1}^{-1} + D\textcircled{1}^{-2}\end{aligned}$$

for some $D \in \mathbb{R}$. Hence

$$\begin{aligned}1 &+ 4\textcircled{1}x_1^*\Big((x_1^*)^2 + (x_2^*)^2 - 2\Big)^3 \\ &= 1 + \Big[4A\textcircled{1} + 4B + 4C\textcircled{1}^{-1}\Big]\Big[\big(2A^2 - 2\big) + 2AB\textcircled{1}^{-1} + D\textcircled{1}^{-2}\Big]^3.\end{aligned} \tag{14}$$

Now if $2A^2 - 2 \neq 0$ there is still a term of the order $\textcircled{1}$ unless $A = 0$. If, instead, $2A^2 - 2 = 0$ a term $\textcircled{1}^{-1}$ can be factored out from the terms in the second parenthesis and the quantity (14) becomes

$$1 + \Big[4A\textcircled{1} + 4B + 4C\textcircled{1}^{-1}\Big]\textcircled{1}^{-3}\Big[+2AB + D\textcircled{1}^{-1}\Big]^3$$

and the finite term cannot be equal to 0. Therefore, in this case, when Constraint Qualification conditions do not hold, the solution of $\nabla F(x) = 0$ does not provide a KKT pair for the constrained problem.

## 2 Quadratic programming

In this section we specialize the results in [1] to the fundamental Quadratic Programming problem

$$\begin{array}{ll} \min\limits_x & \frac{1}{2}x^T M x + q^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \tag{15}$$

where $M \in \mathrm{I\!R}^{n\times n}$ positive definite, $A \in \mathrm{I\!R}^{n\times m}$ with $\mathrm{rank}(A) = m$, $q \in \mathrm{I\!R}^n$, and $b \in \mathrm{I\!R}^m$. We assume that the feasible region is not empty.

For this problem the Karush–Kuhn–Tucher (KKT) conditions [39] are

$$\begin{array}{rcl} Mx + q - A^T u - v & = & 0, \\ Ax - b & = & 0, \\ x & \geq & 0, \\ v & \geq & 0, \\ x^T v & = & 0. \end{array} \tag{16}$$

Following [1], we introduce the unconstrained nonlinear optimization problem

$$\min \frac{1}{2} x^T M x + \frac{\text{①}}{2} \|Ax - b\|_2^2 + \frac{\text{①}}{2} \|\max\{0, -x\}\|_2^2 =: F(x). \tag{17}$$

We have

$$\nabla F(x) = Mx + q + \text{①} A^T (Ax - b) - \text{①} \max\{0, -x\}$$

and a stationary point $x^*$ of (17) satisfies $\nabla F(x^*) = 0$.

The lemma below links stationary points of the penalization function and feasible points for the unconstrained problem.

**Lemma 1** *Let*

$$X := \{x \in \mathrm{I\!R}^n : Ax = b, x \geq 0\}$$

*where* $A \in \mathrm{I\!R}^{n\times m}$ *with* $\mathrm{rank}(A) = m$, *and* $b \in \mathrm{I\!R}^m$. *Assume that* $X \neq \emptyset$. *Let*

$$\psi : x \in \mathrm{I\!R}^n \to \psi(x) = \frac{1}{2} \|Ax - b\|_2^2 + \frac{1}{2} \|\max\{0, -x\}\|_2^2 \in \mathrm{I\!R}.$$

*Then* $\nabla \psi(\bar{x}) = 0$ *if and only if*

$$A\bar{x} - b = 0 \text{ and } \bar{x} \geq 0 \tag{18}$$

Proof. The function $\psi(.)$ is convex and

$$\nabla \psi(x) = A^T (Ax - b) - \max\{0, -x\}.$$

Since $X \neq \emptyset$, it follows that $0 = \min_x \psi(x)$ and, in fact, $x \in X$ if and only if $\psi(x) = 0$.

Moreover, if $\bar{x} \in X$, $A\bar{x} = b$ and $\bar{x} \geq 0$ and hence $\nabla \psi(\bar{x}) = 0$.

Now if $\nabla \psi(\bar{x}) = 0$, from the convexity of the function $\psi(.)$ it follows that

$$\psi(x) \geq \psi(x^*) + \nabla \psi(\bar{x})^T (x - \bar{x}) = \psi(x^*), \quad \forall x \in \mathrm{I\!R}^n.$$

Let now $\hat{x} \in X$, then

$$0 = \psi(\hat{x}) \geq \psi(\bar{x}) \geq 0.$$

Therefore, if $\nabla \psi(\bar{x}) = 0$ it follows that $\psi(\bar{x}) = 0$ and hence $\bar{x} \in X$, that is $A\bar{x} - b = 0$, $\bar{x} \geq 0$.

Let the stationary point $x^*$ be represented, according to the ① new numeral system, in the form (see (4))

$$x^{*,0}①^0 x^{*,1}①^{-1} x^{*,2}①^{-2} \dots$$

that is

$$x^* = x^{*,0} + ①^{-1}x^{*,1} + ①^{-2}x^{*,2} + \dots$$

Then

$$\begin{aligned} 0 \quad = \quad & M\Big(x^{*,0} + ①^{-1}x^{*,1} + ①^{-2}x^{*,2} + \dots\Big)q \\ & + ①A^T\Big(A\left(x^{*,0} + ①^{-1}x^{*,1} + ①^{-2}x^{*,2} + \dots\right) - b\Big) \\ & + ①\max\big\{0, -x^{*,0} - ①^{-1}x^{*,1} - ①^{-2}x^{*,2} + \dots\big\} \end{aligned} \tag{19}$$

From the ① term, it follows that (see Lemma 1)

$$A^T\left(Ax^{*,0} - b\right) + \max\left\{0, -x^{*,0}\right\} = 0$$

from which

$$Ax^{*,0} - b = 0, \tag{20}$$

and

$$\max\left\{0, -x^{*,0}\right\} = 0, \quad \text{that is} \qquad x^{*,0} \geq 0. \tag{21}$$

Looking now at the $①^0$ terms

$$Mx^{*,0} + q + A^T\left(Ax^{*,1} - b^{(1)}\right) - v = 0,$$

where

$$v_j = \max\left\{0, -x^{*,1}\right\}$$

only for the indices $j$ for which $x_j^{*,1} = 0$, otherwise $v_j = 0$.

Setting $u = Ax^{*,1} - b^{(1)}$, from the above formula it follows that

$$Mx^{*,0} + q + A^T u + v = 0 \tag{22}$$

where

$$v_j = \begin{cases} 0 & \text{if } x_j^{*,0} = 0 \\ \max\left\{0, -x_j^{*,1}\right\} & \text{otherwise} \end{cases}$$

Hence

$$v_j \geq 0, \qquad j = 1, \dots, n \tag{23}$$

and

$$v_j x_j^{*,0} = 0, \qquad j = 1, \dots, n. \tag{24}$$

Hence the triplet $\left(x^{*,0}, u, v\right)$ satisfies the KKT conditions (16).

## 3 Conclusions

Exact penalty methods have been largely investigated in the literature for the solution of constrained optimization problems. In this context the main difficulty is that the resulting penalty function is non–smooth (unless additional terms related to first order optimality conditions [40] are introduced, thus making the penalty function much more complicate). Therefore, effective and fast algorithms for minimization of smooth functions cannot be used. However, the introduction of ① allows to define smooth exact penalty functions [1]. The two main contributions of this paper are (1) to underline the importance of Constraints Qualification (whose role is well known in nonlinear optimization theory) even in the case of exact penalty functions using ①, and (2) to specialize the results to the important case of quadratic minimization problems.

Future works include the realization of specific algorithms or the specialization of known ones for the solution of the unconstrained minimization problem (6). In particular, it should be noticed that the separation between infinite, finite and infinitesimal terms in the expression of $F(x)$ allows to immediately determine two (descent) search directions, setting in this way the stage for new algorithms based on two-dimensional search.

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